Claim:

$$[nx] = \sum_{k=0}^{n-1} \left[x + \frac{k}{n}\right]$$

Proof:

$$\left[x + \frac{k}{n}\right] = \begin{cases} [x] & \text{if } x < [x] + \frac{n-k}{n}, \\ [x+1] & \text{if } x \geq [x] + \frac{n-k}{n} \end{cases} \quad ; k \in \mathbb{Z}, 0 \leq k < n \tag{1}$$

$[nx]$ is a step function with

$$P = \left\{ \frac{a}{n} | \quad \forall a \in \mathbb{Z} \right\}$$

.

This motivates us to initially consider only the division containing x:

$$[x] + \frac{j}{n} \ \leq \ x \ < \ [x] + \frac{j+1}{n} \quad ; j \in \mathbb{Z}, 0 \leq j < n \tag{2}$$

This allows us to derive a closed form of the summation:

$$\sum_{k=0}^{n-1} \left[x + \frac{k}{n}\right] = \sum_{k=0}^{n-j-1} \left[x + \frac{k}{n}\right] + \sum_{k=n-j}^{n-1} \left[x + \frac{k}{n}\right] \tag{3}$$

For the first term on the RHS of (3)

$$k \leq n - j - 1 \Rightarrow [x] + \frac{j+1}{n} \leq [x] + \frac{n-k}{n}$$

Applying (2) we get

$$x < [x] + \frac{n-k}{n} \Rightarrow \sum_{k=0}^{n-j-1} \left[x + \frac{k}{n}\right] = \sum_{k=0}^{n-j-1} [x] = (n-j)[x] \tag{4}$$

For the second term on the RHS of (3)

$$k \geq n - j \Rightarrow [x] + \frac{j}{n} \geq [x] + \frac{n-k}{n}$$

Again applying (2) we get

$$x \geq [x] + \frac{n-k}{n} \Rightarrow \sum_{k=n-j}^{n-1} \left[x + \frac{k}{n}\right] = \sum_{k=n-j}^{n-1} [x+1] = j[x+1] \tag{5}$$

So, combining (3), (4) and (5), under condition (2) we have

$$\sum_{k=0}^{n-1} \left[x + \frac{k}{n}\right] = n[x] + j$$

Condition (2) can be rewritten as

$$n[x] + j \ \leq \ nx \ < \ n[x] + j + 1 \quad \Rightarrow \quad [nx] = n[x] + j \tag{6}$$

Thus, for condition (2)

$$\sum_{k=0}^{n-1} \left[x + \frac{k}{n} \right] = [nx]$$

and since j is not involved in the formula, it is true $\forall x$.